

A predictive A_4 model, Charged Lepton Hierarchy and Tri-bimaximal Sum Rule

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Abstract

We propose a novel A_4 model in which the Tri-Bimaximal (TB) neutrino mixing and the charged lepton mass hierarchy are reproduced simultaneously. At leading order, the residual symmetry of the neutrino sector is $Z_2 \times Z_2$ which guarantees the TB mixing without adjusting ad hoc free parameters. In the charged lepton sector, one of the previous Z_2 is maximally broken and the resulting mass matrix is nearly diagonal and hierarchical. A natural mechanism for the required vacuum alignment is given with the help of the supersymmetry and an abelian symmetry factor. In our model, subleading effects which could lead to appreciable deviations from TB mixing are very restrictive giving rise to possible next-to-leading predictions. From an explicit example, we show that our “constrained” A_4 model is a natural framework, based on symmetry principle, to incorporate the TB sum rule: $\sin^2\theta_{12} = 1/3 + 2\sqrt{2} (\cos\delta \sin\theta_{13})/3$.

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1 Introduction

Nowadays continuous improvement on the knowledge of neutrino oscillation parameters makes desirable a neutrino model building going beyond the mere fitting procedure. In particular the leptonic mixing pattern, so different from the one in the quark sector, provides a non-trivial theoretical challenge. The present data [1], at 1σ :

$$\theta_{12} = (34.5 \pm 1.4)^\circ \quad , \quad \theta_{23} = (42.3_{-3.3}^{+5.1})^\circ \quad , \quad \theta_{13} = (0.0_{-0.0}^{+7.9})^\circ \quad , \quad (1)$$

are fully compatible with the so-called Tri-Bimaximal (TB) mixing matrix:

$$U_{\text{TB}} = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & +1/\sqrt{2} \end{pmatrix} \quad , \quad (2)$$

which corresponds to

$$\sin^2 \theta_{12} = \frac{1}{3} \quad (\theta_{12} = 35.3^\circ) \quad , \quad \sin^2 \theta_{23} = \frac{1}{2} \quad , \quad \sin^2 \theta_{13} = 0 \quad . \quad (3)$$

Several interesting ideas leading to a nearly TB mixing have been suggested in the last years [2]. The TB mixing has the advantage of correctly describing the solar mixing angle, which, at present, is the most precisely known. Indeed, its 1σ error, 1.4 degrees corresponds to less than λ_c^2 radians, where $\lambda_c \approx 0.22$ denotes the Cabibbo angle.

TB pattern belongs to the class of mixing textures which are independent on the mass eigenstates. Mass-independent mixing textures usually exhibit an underlying discrete symmetry nature [3]. It has been realized that the TB mixing matrix of Eq. (2) can naturally arise as the result of a particular vacuum alignment of scalars that break spontaneously certain discrete flavour symmetries. A class of very promising models are based on A_4 flavour symmetry [4, 5] and subsequently extended to the group T' [6] to cover a reasonable description also for quarks. Despite of the success, the original A_4 models proposed by Altarelli and Feruglio (AF) in [5] require improvement for various reasons. First of all, the leading order results of AF are affected by a large number of subleading corrections. Even though these corrections are hopefully under control, they are totally independent and the model loses the possibility to go beyond the leading prediction. Furthermore the mass eigenvalues are completely unspecified by A_4 and the charged lepton mass hierarchy can be explained only by an extra Froggatt-Nielsen (FN) [7] $U(1)_{\text{FN}}$ factor. Finally, A_4 alone seems unfavorable to accommodate quark masses and eventually be embedded into a GUT theory, despite some recent attempts in this direction: Pati-Salam [8], $SU(5)$ [9, 10], $SO(10)$ [11]. Indeed, it is a quite non trivial task to reproduce all charged fermion hierarchies compatible with a natural mechanism for the vacuum alignment. Particular efforts have been made in this direction. For example, in [12], the problem of fermion hierarchy is partially solved by embedding A_4 into a continuous left-right symmetry. However the question of a natural vacuum alignment in their model remains open. In the recent proposal of A_4 model in a 5D SUSY $SU(5)$ GUT [10], the fermion mass hierarchies and mixing are a result of the interplay of three different sources: a discrete flavour group based on A_4 , wave-function suppressions of bulk fields and an additional $U(1)_{\text{FN}}$.

In the literature, there are many mechanisms based on different discrete groups that can successfully describe TB mixing at leading order. Another important issue for the model building is if there were some criteria to distinguish various constructions. One

possibility is to go beyond the leading order prediction. There is a remarkable sum rule in the lepton mixing sector [13]:

$$\sin^2\theta_{12} = \sin^2\theta_{12}^\nu + \sin^2 2\theta_{12}^\nu \cos\delta \sin\theta_{13} , \quad (4)$$

where δ is the Dirac CP violation phase. This sum rule can be derived when maximal θ_{23} and $\theta_{13} = 0$ come from the neutrino sector at leading order and a non vanishing θ_{13} arises only when one includes small charged lepton mixing (under certain assumptions such as θ_{12}^e or θ_{13}^e dominance as will be explained in the section 5.1). In flavour models in which $\sin^2\theta_{12}^\nu = 1/3$ holds almost exactly, this sum rule should be a precise test. However, generally, in the dynamical realization of TB mixing pattern based on discrete flavour symmetries, the Eq. (4) is not precisely predicted. For example, in the context of the AF models, higher order corrections generically contribute to neutrino masses as well as to the charged lepton masses, all of relative order λ_c^2 , being λ_c the Cabibbo angle. Then one could only expect that $\sin^2\theta_{12}^\nu = 1/3 + O(\lambda_c^2)$ and $\sin^2\theta_{13} = O(\lambda_c^2)$, if the model is, in some sense, “unconstrained”.

In this work, we will give a realization of the TB pattern solving the charged lepton hierarchy problem and discuss a possible subleading prediction according to Eq. (4). The model is supersymmetric and based only on a discrete symmetry $A_4 \times G$ where G is an abelian factor. The mixing matrix of Eq. (2) is obtained in the neutrino sector by the spontaneous A_4 breaking. Supersymmetry (SUSY) is introduced to simplify the discussion of the vacuum alignment. At the lowest order of the expansion parameters $\langle\varphi\rangle/\Lambda \ll 1$, only the tau mass is generated together with the TB mixing. The muon and electron masses are subsequently generated by higher orders of the same expansion, similar to what happens in the recently proposed S_3 model [14]. This is one of the distinguished features of our model: the charged lepton hierarchy is also controlled by the spontaneous breaking of A_4 without introducing an extra $U(1)_{FN}$ factor. The abelian factor G is given by $Z_3 \times Z_3'^1$. The presence of an abelian factor G is essential for our construction. First of all, G guarantees the misalignment in flavour space between the neutrino and the charged lepton mass eigenstates, responsible for both TB mixing and charged lepton hierarchy. Furthermore, G plays an important role in suppressing subleading contributions in order to keep the model predictive. From this point of view, the present A_4 model is more constrained than the other A_4 models and we will refer it as a constrained A_4 model. TB mixing in the neutrino sector holds almost exactly and subleading corrections generically lead to interesting correlations between parameters. In particular, corrections to the charged lepton sector are subjected by the neutrino sum rule of Eq. (4) and offer a possible test of our model.

In section 2 we outline the main features of our model focusing on the symmetry breaking pattern. We then move to solve explicitly the vacuum alignment problem in a SUSY context in section 3 finding a new type of minimum of the scalar potential. In section 4 we construct a simple model of leptons with the symmetry breaking pattern according to the vacuum alignment. In section 5, we will discuss possible predictive deviations from the TB pattern, in particular those related to the sum rule of Eq. (4). Finally, in section 6, we comment on other aspects of our model and then conclude.

¹It is worth to keep in mind that the abelian factor Z_3 or Z_3' does not correspond to a discrete component of $U(1)_{FN}$. As we will see they are not directly responsible for the charged lepton hierarchy.

2 The ingredients for an alternative realization of A_4

The group A_4 has 12 elements and four non-equivalent irreducible representations: one triplet and three independent singlets 1, $1'$ and $1''$. Elements of A_4 are generated by the two generators S and T obeying the relations:

$$S^2 = (ST)^3 = T^3 = 1 \quad . \quad (5)$$

We will consider the following unitary representations of T and S :

$$\begin{aligned} \text{for } 1 \quad S &= 1 \quad T = 1 \\ \text{for } 1' \quad S &= 1 \quad T = e^{i4\pi/3} \equiv \omega^2 \\ \text{for } 1'' \quad S &= 1 \quad T = e^{i2\pi/3} \equiv \omega \end{aligned} \quad (6)$$

and for the triplet representation

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \quad . \quad (7)$$

The tensor product of two triplets is given by $3 \times 3 = 1 + 1' + 1'' + 3_S + 3_A$. From (6) and (7), one can easily construct all multiplication rules of A_4 . In particular, for two triplets $\psi = (\psi_1, \psi_2, \psi_3)$ and $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ one has:

$$\begin{aligned} \psi_1\varphi_1 + \psi_2\varphi_3 + \psi_3\varphi_2 &\sim 1 \ , \\ \psi_3\varphi_3 + \psi_1\varphi_2 + \psi_2\varphi_1 &\sim 1' \ , \\ \psi_2\varphi_2 + \psi_3\varphi_1 + \psi_1\varphi_3 &\sim 1'' \ , \\ \begin{pmatrix} 2\psi_1\varphi_1 - \psi_2\varphi_3 - \psi_3\varphi_2 \\ 2\psi_3\varphi_3 - \psi_1\varphi_2 - \psi_2\varphi_1 \\ 2\psi_2\varphi_2 - \psi_1\varphi_3 - \psi_3\varphi_1 \end{pmatrix} &\sim 3_S \ , \quad \begin{pmatrix} \psi_2\varphi_3 - \psi_3\varphi_2 \\ \psi_1\varphi_2 - \psi_2\varphi_1 \\ \psi_3\varphi_1 - \psi_1\varphi_3 \end{pmatrix} \sim 3_A \ . \end{aligned} \quad (8)$$

The group A_4 has two obvious subgroups: G_S , which is a reflection subgroup generated by S and G_T , which is the group generated by T , isomorphic to Z_3 .

First of all, we will quickly revisit the original proposal of AF based on $A_4 \times Z_3$. Lepton $SU(2)$ doublets l_i ($i = e, \mu, \tau$) are assigned to the triplet A_4 representation, while the lepton singlets e^c , μ^c and τ^c are assigned to 1, $1''$ and $1'$, respectively. The symmetry breaking sector consists of the scalar fields neutral under the SM gauge group: $(\varphi_T, \varphi_S, \xi)$, transforming as $(3, 3, 1)$ of A_4 . The additional Z_3 discrete symmetry is needed in order to reproduce the desired alignment and for the separation of charged lepton and neutrino sectors. The key feature of their model is that the minimization of the scalar potential at the leading order leads to the following VEVs:

$$\langle \varphi_T \rangle \propto (1, 0, 0) \ , \quad \langle \varphi_S \rangle \propto (1, 1, 1) \ , \quad \langle \xi \rangle \neq 0 \quad . \quad (9)$$

In the charged lepton sector the flavour symmetry A_4 is broken by $\langle \varphi_T \rangle$ down to $G_T \simeq Z_3$ generated by T . At leading order in $1/\Lambda$, charged lepton masses are diagonal simply because there is a low-energy G_T symmetry. In the neutrino sector A_4 is broken down to $G_S \simeq Z_2$ generated by S . The absence of the scalar singlets $1'$ and $1''$ in the

neutrino sector implies that the resultant neutrino mass matrix is the most general one allowed by $G_S \times G_{2-3}$ where G_{2-3} is generated by

$$S_{2-3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

$G_{2-3} \simeq Z_2$ is the permutation symmetry of the second and third generation of neutrinos. It is by now known that the TB structure of lepton mixing is a result of the intrinsic $G_S \times G_{2-3}$ symmetry in the basis where the charged leptons are diagonal [3].

In our alternative realization of A_4 model, the neutrino sector is unchanged at leading order. The only difference is that the TB mixing pattern results almost exact (its precise meaning will be clear later) even including subleading corrections. For the charged leptons, we would like to investigate the consequence of an alternative alignment of φ_T :

$$\langle \varphi_T \rangle \propto (0, 1, 0) \quad (10)$$

which entirely breaks A_4 . Such a VEV breaks also the permutation symmetry in a maximal way, since

$$\langle \varphi_T \rangle^t S_{2-3} \langle \varphi_T \rangle = 0.$$

From the group theory point of view, there is an essential difference between the alignment (9) and that considered in the present paper (10). The first one is a subgroup-preserving direction, while the second one breaks A_4 down to nothing. Observe that a natural realization of (9) in the AF model requires that the scalar potential for φ_T and (φ_S, ξ) is actually separated. This can be done, for example, by imposing different abelian charges for φ_T and (φ_S, ξ) under Z_3 . Now consider a more general scalar potential with only one triplet φ_T plus eventually other singlets of A_4 : $V(\varphi_T, \dots)$. Interestingly, we find that leading order SUSY vacua of the type (10) can never be obtained without fine-tuning if $V(\varphi_T, \dots)$ is symmetric only with respect to A_4 (without considering additional abelian factor G). An alignment of the type (10) that entirely breaks A_4 must be highly fine tuned. For this reason, it is of general interest to find conditions under which (10) could be obtained naturally. The strategy is to enlarge the A_4 group by an additional abelian factor G . The desired alignment can then be regarded as a subgroup-preserving direction of $A_4 \times G$.

Consider an enlarged flavour group $A_4 \times Z'_3$ for charged leptons. Now the alignment (10) preserves the subgroup $G_{T+} \simeq Z_3$ generated by T^+ defined as a simultaneous transformation of $T \in A_4$ and $\omega \in Z'_3$, or briefly

$$T^+ = \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}. \quad (11)$$

In addition to G_{T+} , $A_4 \times Z'_3$ has another subgroup isomorphic to Z_3 , not contained in A_4 : G_{T-} generated by T^- defined as a simultaneous transformation of T and ω^2 , or briefly

$$T^- = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (12)$$

Hereafter, we will use G_T for the subgroup $G_T \times 1$ of $A_4 \times Z'_3$. As we will see in a moment, from the point of view of $A_4 \times Z'_3$, the alignment considered in our paper has rather

different implications for charged lepton masses in comparison to the original AF model based on A_4 .

The difference between $\langle\varphi_T\rangle \propto (1, 0, 0)$ and $\langle\varphi_T\rangle \propto (0, 1, 0)$ can be seen in another way. From the tensor product given in (8), we see that, in the first case, $\langle\varphi_T\rangle^m$ for any integer $m \geq 1$ is aligned in the same direction of $\langle\varphi_T\rangle$. Instead, in the second case, this does not happen. In fact, $\langle\varphi_T\rangle \propto (0, 1, 0)$ implies that $\langle\varphi_T\rangle^{3n+1} \propto (0, 1, 0)$, $\langle\varphi_T\rangle^{3n+2} \propto (0, 0, 1)$, $\langle\varphi_T\rangle^{3n+3} \propto (1, 0, 0)$. These three directions preserve respectively G_{T+} , G_{T-} and G_T , subgroups of $A_4 \times Z'_3$ defined before. Considering the case $n = 0$, the three orthogonal directions generated by $\langle\varphi_T\rangle$, $\langle\varphi_T\rangle^2$ and $\langle\varphi_T\rangle^3$ are, in our model, responsible for the charged lepton hierarchy. Assigning the lepton doublets l_i to the triplet A_4 representation and the lepton singlets $e^c, \mu^c, \tau^c \sim 1$, we have the most general mass matrices invariant under G_{T+} , G_{T-} and G_T respectively (we will use a convention in which the conjugate fields lie on the left hand side of superpotentials):

$$m_l^{(1)} = \begin{pmatrix} 0 & 0 & \times \\ 0 & 0 & \times \\ 0 & 0 & \times \end{pmatrix}, \quad m_l^{(2)} = \begin{pmatrix} 0 & \times & 0 \\ 0 & \times & 0 \\ 0 & \times & 0 \end{pmatrix}, \quad m_l^{(3)} = \begin{pmatrix} \times & 0 & 0 \\ \times & 0 & 0 \\ \times & 0 & 0 \end{pmatrix}. \quad (13)$$

Since the masses $m_l^{(1)}$, $m_l^{(2)}$ and $m_l^{(3)}$ arise at order $\langle\varphi_T\rangle/\Lambda$, $(\langle\varphi_T\rangle/\Lambda)^2$ and $(\langle\varphi_T\rangle/\Lambda)^3$ respectively, we automatically obtain the correct hierarchy between the charged lepton masses, $m_e \ll m_\mu \ll m_\tau$, if we assume $\langle\varphi_T\rangle/\Lambda \sim \lambda_c^2$, being λ_c the Cabibbo angle.

In the next section, we will explicitly solve the vacuum alignment problem in the enlarged flavour group, but now we will give an insight into some important features of our model. The full flavour symmetry is based on $A_4 \times Z'_3 \times Z_3$ where A_4 is responsible for the TB lepton mixing and the abelian factor $Z'_3 \times Z_3$ is important in the vacuum alignment analysis. Furthermore, $Z'_3 \times Z_3$ suppresses almost any interactions between the neutrino and the charged lepton sectors. The fields of the model, together with their transformation properties under the flavour group, are listed in Table 1. In the neutrino sector, A_4 is broken down to G_S exactly as in the AF model. For charged leptons, both A_4 and Z'_3 are broken, however, their “diagonal” combination G_{T+} survives at leading order. The breaking of $A_4 \times Z'_3$ by $\langle\varphi_T\rangle/\Lambda$, $(\langle\varphi_T\rangle/\Lambda)^2$ and $(\langle\varphi_T\rangle/\Lambda)^3$ generates, as discussed before, a hierarchical structure of the charged lepton mass matrix m_l . However, if m_l were $m_l^{(1)} + m_l^{(2)} + m_l^{(3)}$, the left handed mixings would be small enough ($\sim O(\lambda_c^2)$), whereas large right-handed mixings necessarily would arise. This structure of m_l is disfavored by a future GUT embedding based on $SU(5)$, since the relation $m_l \sim m_d^T$ would imply large mixings in the down quark sector. In general one should not expect that the large down quark mixings could be cancelled by large up quark mixings in the context of $SU(5)$. This potential problem should not be present if we assign different Z'_3 charges for lepton singlets in such a way that τ^c couples with φ_T , μ^c with φ_T^2 and e^c with φ_T^3 as shown in Table 1. In fact, we will arrive at a diagonal and hierarchical m_l :

$$m_l = \begin{pmatrix} \varphi_T^3/\Lambda^3 & 0 & 0 \\ 0 & \varphi_T^2/\Lambda^2 & 0 \\ 0 & 0 & \varphi_T/\Lambda \end{pmatrix} v_d. \quad (14)$$

3 Vacuum Alignment

In this section we will discuss the minimization of the scalar potential. To achieve the desired alignment in a simple way, we work with a supersymmetric model, with $N=1$

Field	1	e^c	μ^c	τ^c	h_u	h_d	φ_T	ξ'	φ_S	ξ	$\tilde{\xi}$	φ_0^T	φ_0^S	ξ_0
A_4	3	1	1	1	1	1	3	1'	3	1	1	3	3	1
Z_3	ω	1	1	1	1	ω^2	1	1	ω	ω	ω	1	ω	ω
Z'_3	1	ω	ω^2	1	1	ω^2	ω	ω	1	1	1	ω	1	1
$U(1)_R$	1	1	1	1	0	0	0	0	0	0	0	2	2	2

Table 1: The transformation properties of leptons, electroweak Higgs doublets and flavons under $A_4 \times Z_3 \times Z'_3$ and $U(1)_R$.

SUSY, eventually broken by small soft breaking terms. In association with φ_T we introduce another scalar field $\xi' \sim 1'$ under A_4 . In general, it is not easy to realize non-trivial minima for flavon fields preserving different subgroups of A_4 . The key point is that the abelian part of the discrete symmetry G should forbid unwanted terms in the driving superpotential. In our case, the charged lepton hierarchy is generated by $\langle \varphi_T \rangle \propto (0, 1, 0)$ which breaks entirely A_4 . In the neutrino sector, the TB mixing pattern requires $\langle \varphi_S \rangle \propto (1, 1, 1)$ and $\langle \xi \rangle \neq 0$ which preserve a Z_2 subgroup of A_4 . We find that the minimal choice of G in order to obtain the required vacuum alignment is given by $Z_3 \times Z'_3$.

Apart from the discrete symmetry group $A_4 \times G$ the superpotential w is automatically invariant also under a continuous $U(1)_R$ symmetry under which matter fields have $R = +1$, while Higgses and flavons have $R = 0$. Such a symmetry will be eventually broken down to R-parity by small SUSY breaking effects that can be neglected in the first approximation in our analysis. The spontaneous breaking of A_4 can be employed by introducing a new set of multiplets, the driving fields, with $R = 2$. We introduce a driving field ξ_0 , fully invariant under A_4 , and two driving fields φ_0^T and φ_0^S , triplets of A_4 . As reported in Table 1, ξ_0 and φ_0^S are charged under Z_3 which are responsible for the alignment of φ_S , while φ_0^T is charged under Z'_3 which drives a non-trivial VEV of φ_T . The most general driving superpotential w_d invariant under $A_4 \times G$ with $R = 2$ is given by

$$w_d = g_1(\varphi_0^S \varphi_S \varphi_S) + g_2 \tilde{\xi}(\varphi_0^S \varphi_S) + g_3 \xi_0(\varphi_S \varphi_S) + g_4 \xi_0 \xi^2 + g_5 \xi_0 \tilde{\xi} + g_6 \xi_0 \tilde{\xi}^2 \quad (15)$$

$$+ h_1 \xi'(\varphi_0^T \varphi_T)'' + h_2(\varphi_0^T \varphi_T \varphi_T). \quad (16)$$

Terms with φ_S, φ_T interchanged are forbidden by $Z_3 \times Z'_3$. It is important to observe that the absence of $M(\varphi_0^T \varphi_T)$ is essential for the desired vacuum alignment of φ_T . Eq. (15) and Eq. (16) give two decoupled sets of F-terms for driving fields which characterize the supersymmetric minimum. From Eq. (15) we have:

$$\begin{aligned} \frac{\partial w}{\partial \varphi_{01}^S} &= g_2 \tilde{\xi} \varphi_{S1} + 2g_1(\varphi_{S1}^2 - \varphi_{S2} \varphi_{S3}) = 0 \\ \frac{\partial w}{\partial \varphi_{02}^S} &= g_2 \tilde{\xi} \varphi_{S3} + 2g_1(\varphi_{S2}^2 - \varphi_{S1} \varphi_{S3}) = 0 \\ \frac{\partial w}{\partial \varphi_{03}^S} &= g_2 \tilde{\xi} \varphi_{S2} + 2g_1(\varphi_{S3}^2 - \varphi_{S1} \varphi_{S2}) = 0 \\ \frac{\partial w}{\partial \xi_0} &= g_4 \xi^2 + g_5 \xi \tilde{\xi} + g_6 \tilde{\xi}^2 + g_3(\varphi_{S1}^2 + 2\varphi_{S2} \varphi_{S3}) = 0 \end{aligned} \quad (17)$$

We can enforce $\langle \tilde{\xi} \rangle = 0$ ² by adding to the scalar potential a soft SUSY breaking mass

²Since there is no fundamental distinction between the singlets ξ and $\tilde{\xi}$ we have defined $\tilde{\xi}$ as the combination that couples to $(\varphi_0^S \varphi_S)$ in the superpotential w_d . The introduction of an additional singlet is essential to recover a non-trivial solution.

term for the scalar field $\tilde{\xi}$, with $m_{\tilde{\xi}}^2 > 0$. In this case, in a finite portion of the parameter space, we find the solution

$$\begin{aligned}\langle \tilde{\xi} \rangle &= 0, & \langle \xi \rangle &= u, \\ \langle \varphi_S \rangle &= (v_S, v_S, v_S), & v_S^2 &= -\frac{g_4}{3g_3}u^2,\end{aligned}\tag{18}$$

with u undetermined, which is exactly analogue to the AF model. Setting to zero the F-terms from Eq. (16), we have:

$$\begin{aligned}\frac{\partial w}{\partial \varphi_{01}^T} &= h_1 \xi' \varphi_{T3} + 2h_2(\varphi_{T1}^2 - \varphi_{T2} \varphi_{T3}) = 0 \\ \frac{\partial w}{\partial \varphi_{02}^T} &= h_1 \xi' \varphi_{T2} + 2h_2(\varphi_{T2}^2 - \varphi_{T1} \varphi_{T3}) = 0 \\ \frac{\partial w}{\partial \varphi_{03}^T} &= h_1 \xi' \varphi_{T1} + 2h_2(\varphi_{T3}^2 - \varphi_{T1} \varphi_{T2}) = 0\end{aligned}$$

and the solution to these four equations is:

$$\langle \xi' \rangle = u' \neq 0, \quad \langle \varphi_T \rangle = (0, v_T, 0), \quad v_T = -\frac{h_1 u'}{2h_2}.\tag{19}$$

with u' undetermined. The flat directions can be removed by the interplay of radiative corrections to the scalar potential and soft SUSY breaking terms for ξ' , with $m_{\xi'}^2 < 0$. It is worth to observe that since the two abelian factors in $Z'_3 \times Z_3$ are complementary for (φ_T, ξ') and (φ_S, ξ) , the VEV alignments (18) and (19) are independent up to $1/\Lambda^2$.

4 A $A_4 \times Z_3 \times Z'_3$ model for leptons

In this section we propose a very simple SUSY model for leptons based on the following pattern of symmetry breaking of $A_4 \times Z'_3 \times Z_3$

$$\begin{aligned}\langle \varphi_T \rangle &= (0, v_T, 0), & \langle \varphi_S \rangle &= (v_S, v_S, v_S) \\ \langle \xi \rangle &= u, & \langle \tilde{\xi} \rangle &= 0, & \langle \xi' \rangle &= u' .\end{aligned}\tag{20}$$

In the charged lepton sector the flavour symmetry $A_4 \times Z'_3$ is broken by (φ_T, ξ') down to G_{T+} (with Z_3 unbroken) at leading order where only the tau mass is generated. The muon and electro masses are generated by higher order contributions. In the neutrino sector $A_4 \times Z_3$ is broken by (φ_S, ξ) down to G_S (with Z'_3 unbroken) with an accidental extra G_{2-3} symmetry as noted benfore.

Precisely, the lepton masses are described by w_l , given by, up to $1/\Lambda^3$:

$$\begin{aligned}w_l &= \alpha_1 \tau^c (l \varphi_T) h_d / \Lambda \\ &+ \beta_1 \mu^c \xi' (l \varphi_T)'' h_d / \Lambda^2 + \beta_2 \mu^c (l \varphi_T \varphi_T) h_d / \Lambda^2 \\ &+ \gamma_1 e^c (\xi')^2 (l \varphi_T)' h_d / \Lambda^3 + \gamma_2 e^c \xi' (l \varphi_T \varphi_T)'' h_d / \Lambda^3 + \gamma_3 e^c (l \varphi_T \varphi_T \varphi_T) h_d / \Lambda^3 \\ &+ \gamma' e^c (l \varphi_S) \xi^2 h_d / \Lambda^3 + \gamma'' e^c (l \varphi_S \varphi_S \varphi_S) h_d / \Lambda^3 \\ &+ (x_a \xi + \tilde{x}_a \tilde{\xi})(ll) h_u h_u / \Lambda^2 + x_b (\varphi_S ll) h_u h_u / \Lambda^2 + \dots\end{aligned}\tag{21}$$

where the forth line is due to the interaction between charged lepton and neutrino sectors. After electroweak symmetry breaking, $\langle h_{u,d} \rangle = v_{u,d}$, given the specific orientation of

$\langle \varphi_T \rangle \propto (0, 1, 0)$, the first three lines in w_l give rise to diagonal and hierarchical mass terms for charged leptons. The only off diagonal contributions to the charged lepton mass matrix come from the fourth line in Eq. (21) after electroweak and flavour symmetry breakings. The off diagonal entries can be however rotated away by small unobservable right-handed rotations (almost of order $O(\text{VEV}/\Lambda)$) without affection eigenvalues of m_l up to $1/\Lambda^3$ and we will not consider these terms in the following. Defining the expansion parameter $v_T/\Lambda \equiv \lambda^2 \ll 1$ and the Yukawa couplings y_l ($l = e, \mu, \tau$) as

$$\begin{aligned} y_\tau &= \alpha_1, \\ y_\mu &= (\beta_1 u'/v_T + 2\beta_2)\lambda^2, \\ y_e &= (\gamma_1(u'/v_T)^2 - \gamma_2 u'/v_T - 2\gamma_3 + \gamma' v_S u^2/v_T^3)\lambda^4, \end{aligned}$$

the charged lepton masses are given by

$$m_l = y_l \lambda^2 v_d \quad (l = e, \mu, \tau) \quad . \quad (22)$$

Here we assume that all VEVs of flavon fields are of the same order of magnitude. Differently from the original proposal of A_4 [4, 5] model or T' model of [6], we are able to produce the required hierarchy among m_e , m_μ and m_τ provided $\lambda \approx \lambda_c$ where λ_c is the Cabibbo angle. Some observations are in order. The different assignment of lepton singlets e^c , μ^c and τ^c under Z'_3 needs to compensate the Z'_3 charge of φ_T . As a result, we obtain automatically a diagonal m_l . However the alignment (10) is, in our model, the true fundamental ingredient in generating the charged lepton hierarchy. As already mentioned, the two abelian factors $Z'_3 \times Z_3$ are complementary for the charged lepton and neutrino sectors. This means that the interaction between the two sectors can arise only at a relative order $1/\Lambda^3$. The only field which is charged under both abelian factors is h_d . As we will explain later on, the previous almost diagonal structure of charged leptons might receive important off-diagonal contributions only when additional breaking effects of A_4 are added.

At this order, all the information about lepton mixing angles is encoded in the neutrino mass matrix which is identical to one of the original A_4 model and T' model:

$$m_\nu = \frac{v_u^2}{\Lambda} \begin{pmatrix} a+2b & -b & -b \\ -b & 2b & a-b \\ -b & a-b & 2b \end{pmatrix} \quad , \quad (23)$$

where

$$a \equiv x_a \frac{u}{\Lambda} \quad , \quad b \equiv x_b \frac{v_S}{\Lambda} \quad . \quad (24)$$

The neutrino mass matrix is diagonalized by the transformation:

$$U^T m_\nu U = \frac{v_u^2}{\Lambda} \text{diag}(a+3b, a, -a+3b) \quad , \quad (25)$$

with $U = U_{\text{TB}}$. Therefore the TB mixing of eq. (2) is reproduced, at the leading order. For the neutrino masses we obtain:

$$\begin{aligned} |m_1|^2 &= \left[-r + \frac{1}{8 \cos^2 \Delta (1-2r)} \right] \Delta m_{\text{atm}}^2 \\ |m_2|^2 &= \frac{1}{8 \cos^2 \Delta (1-2r)} \Delta m_{\text{atm}}^2 \\ |m_3|^2 &= \left[1 - r + \frac{1}{8 \cos^2 \Delta (1-2r)} \right] \Delta m_{\text{atm}}^2 \quad , \end{aligned} \quad (26)$$

where $r \equiv \Delta m_{sol}^2 / \Delta m_{atm}^2 \equiv (|m_2|^2 - |m_1|^2) / (|m_3|^2 - |m_1|^2)$, $\Delta m_{atm}^2 \equiv |m_3|^2 - |m_1|^2$ and Δ is the phase difference between the complex numbers a and b . The value of $|m_{ee}|$, the parameter characterizing the violation of total lepton number in neutrinoless double beta decay, is given by:

$$|m_{ee}|^2 = \left[-\frac{1+4r}{9} + \frac{1}{8 \cos^2 \Delta (1-2r)} \right] \Delta m_{atm}^2 \quad . \quad (27)$$

Independently from the value of the unknown phase Δ we get the relation:

$$|m_3|^2 = |m_{ee}|^2 + \frac{10}{9} \Delta m_{atm}^2 \left(1 - \frac{r}{2} \right) \quad , \quad (28)$$

which is another prediction of our model. The results on light neutrinos coincide with those obtained at leading order in the A_4 and T' models [5, 6]. The new feature of our model is that the enlarged abelian symmetry G strongly suppresses possible higher order contributions and the prediction of the TB mixing pattern and Eq. (28) is (almost, i.e. up to terms with relative suppression of order $1/\Lambda^2$) exact.

5 Deviations from TB mixing

The results of the previous sections hold almost exactly. As already pointed out, the $Z_3 \times Z'_3$ charge assignments in Table 1 forbid all next-to-leading and next-to-next-to-leading order corrections. The non-zero higher order corrections arise only at the relative order $1/\Lambda^3$ with respect to the terms already considered in w_d and w_l . In particular, our model predicts the almost exact TB mixing in the neutrino sector. From the phenomenological point of view, it is also interesting to explore a natural mechanism to generate sizable deviations from TB mixing without loosing the predictivity.

In the literature, model independent approaches to parametrize small corrections to the TB mixing pattern have been widely developed. We should not re-analyze fully all these issues in our model. Particular emphasis concerns special patterns of TB breaking effects, both for the neutrino and the charged lepton sectors, that can lead to testable correlations between parameters [13]. Here we will focus on the so called neutrino mixing sum rule given by Eq. (4) mentioned in the introduction. The goal is to provide an existence proof that predictive deviations from TB mixing can be generated naturally in our constrained A_4 model.

5.1 θ_{13}^e (θ_{12}^e) dominance

One of the most interesting deviations from TB mixing is called θ_{13}^e (θ_{12}^e) dominance [13]. This pattern is realized by assuming that, in the neutrino sector, $U_\nu = U_{TB}$ holds exactly and the only corrections come from a non trivial U_e . In this case, the unitary matrix U_e , parametrized in the standard way, shall involve three small rotations θ_{12}^e , θ_{13}^e , θ_{23}^e and in general three phases. From $U_{PMNS} = U_e^\dagger U_\nu$, one can show that, at first order, the resulting θ_{13} and θ_{12} depend only on θ_{13}^e , θ_{12}^e and δ , a combination of the phases which appear in U_e (for more detail, see [13]). If the θ_{13}^e (θ_{12}^e) contribution dominates over the θ_{12}^e (θ_{13}^e) one, in the literature referred also as θ_{13}^e (θ_{12}^e) dominance, one obtains the following TB mixing sum rule:

$$\sin^2 \theta_{12} = \frac{1}{3} + \frac{2\sqrt{2}}{3} \cos \delta \sin \theta_{13} \quad . \quad (29)$$

To my present knowledge, a model realization of the previous prediction, based on symmetry principle, has never been considered before. Indeed, (29) is not a natural prediction of the original AF model in which $\theta_{12}^e \sim \lambda^2$ and $\theta_{13}^e \sim \lambda^2$ are generated at the same time, together with the subleading contributions to the neutrino sector, all of order λ^2 . Differently, in our constrained A_4 model, an exact TB mixing in the neutrino sector with a diagonal and hierarchical charged lepton mass matrix is predicted without corrections (up to $1/\Lambda^2$). Only additional scalar fields should account for deviations from TB mixing. We will propose a simple extension of our model in which, for example, the θ_{13}^e dominance pattern arises from spontaneous breaking of A_4 with an additional A_4 singlet.

5.2 θ_{13}^e dominance in the constrained A_4 model

The leading order results of our model can be slightly modified by introducing another singlet χ , invariant under Z_3 but carrying a charge ω^2 under Z_3' . As shown in the Appendix A, if χ acquires a large VEV $\langle \chi \rangle = v \sim u'$, it can lead to a sizable correction to $\langle \varphi_T \rangle = v_T(0, 1, 0)$. At the first order in $1/\Lambda$, only the first component of $\langle \varphi_T \rangle$ receives the correction. This is a new feature of the constrained A_4 model. As we shall see in a moment, this result is essential to reproduce the θ_{13}^e dominance pattern³.

Now we begin to discuss the effect of χ on lepton masses and mixing. The introduction of χ slightly modifies w_l of Eq. (21):

$$w_l \rightarrow w_l + \beta_3 e^c \chi (l \varphi_T) h_d / \Lambda^2 + \gamma_4 \mu^c \chi^2 (l \varphi_T) h_d / \Lambda^3 \quad . \quad (30)$$

Taking into account these new terms in the superpotential and the subleading correction to the VEV of φ_T the charged lepton mass matrix becomes (up to $1/\Lambda^3$):

$$m_l = \begin{pmatrix} O(y_e) & 0 & \beta_3 \lambda^2 \\ 0 & y_\mu & O(y_e) \\ \alpha_1 \delta v_T / (\Lambda \lambda^2) & 0 & y_\tau \end{pmatrix} \lambda^2 v_d \quad , \quad (31)$$

where we explicitly give only terms of order up to λ^4 . In our notation, the transformation needed to diagonalize m_l is:

$$V_e^T m_l U_e = \text{diag}(m_e, m_\mu, m_\tau) \quad (32)$$

and the unitary matrix U_e involves rotations of order $\theta_{12}^e = 0$, $\theta_{13}^e = O(\lambda^2)$, $\theta_{23}^e = 0$. Recalling that the neutrino mass matrix m_ν does not receive corrections up to those of relative order $O(\lambda^6)$, U_ν is given by Eq. (2) almost exactly. Then we realize naturally the θ_{13}^e dominance pattern without adjusting adimensional coefficients. Indeed, in this simple extension, we not only reproduce the sum rule (29) but also

$$\sin^2 \theta_{13} = O(\lambda^2) \quad .$$

The subleading effect considered in this section is very important in order to incorporate CP violation in the TB pattern. The smallness of θ_{13} and CP violation are correlated in the θ_{13}^e dominance pattern [13]. The size of CP violation in the neutrino oscillations is measured by the following Jarlskog invariant [16]:

$$J_{\text{CP}} = \text{Im}\{U_{e1}U_{\mu 2}U_{e2}^*U_{\mu 1}^*\} \quad ,$$

³If χ were $\chi \sim (1', 1, \omega^2)$ under $A_4 \times Z_3 \times Z_3'$, we would cover the θ_{12}^e dominance pattern.

depending on the Dirac phase δ . In our case, we obtain a very simple expression: $J_{CP} \approx \frac{1}{6}\lambda^2 \sin\delta$. Since $\sin^2\theta_{12}$ is very close to $1/3$, a value of θ_{13} close to the existing upper limit, necessarily requires a large CP violation. However, without accidental enhancement, we typically have a small θ_{13} , say at a level of $O(\lambda_c^2)$. In this case, the value of δ is very sensible to the closeness of $\sin^2\theta_{12}$ to its tri-bimaximal value. If a sizable CP violation is determined by long-baseline neutrino oscillation experiments, we predict a $\sin^2\theta_{12}$ to be extremely close to $1/3$. Conversely, if a future high precision determination of $\sin^2\theta_{12}$ will come extremely close to $1/3$, we should expect an observable CP asymmetry in long-baseline neutrino oscillation experiments.

6 Further discussions and conclusion

Both charged fermion mass hierarchies and large lepton mixings can be potentially achieved via spontaneous breaking of the flavour symmetry. However, in most cases, the flavour group is of the type $D \times U(1)_{FN}$ where D is a discrete component that controls the mixing angles and $U(1)_{FN}$ is an abelian continuous symmetry that describes the mass hierarchies. It would be a very attractive task to construct economical and constrained models where the same flavon fields producing the mixing pattern by VEV misalignment are also responsible for the mass hierarchies. Models of this type based on the gauged flavour groups $SU(3)$, $SO(3)$ [17] and the non-abelian finite group $PSL_2(7)$ [18] exist in the literature. However in these models the charged fermion mass hierarchies are obtained by a complicated flavour symmetry breaking sector and/or an ad hoc adjustment of the messenger scales. A first simple model belonging to this class has been constructed in [14] based on the minimal non-abelian group S_3 . With a very economical flavon sector, the model is able to provide a decent description of the main features of all the fermion mass spectrum, including the approximate vanishing of θ_{13} and of $\theta_{23} - \pi/4$.

In the present work, we addressed a possible unified picture of the charged lepton mass hierarchy and the Tri-Bimaximal mixing pattern. The model is based on the spontaneously broken A_4 flavour symmetry with an abelian factor given by $Z_3 \times Z'_3$. The very special structure of the leptonic mixings is understood by a mechanism of vacuum misalignment in flavour space. A new type of SUSY minima of the scalar potential is explicitly given. In the neutrino sector, the A_4 component is broken to $G_S \times G_{2-3}$ (where G_{2-3} is an accidental symmetry) as in the original AF models guaranteeing the TB mixing. On the other hand, in the charged lepton sector, A_4 is entirely broken already at the leading order of the $\langle\varphi\rangle/\Lambda$ expansion. A noticeable feature of the model is that the charged lepton hierarchy is also determined by the symmetry breaking of the A_4 group without the use of an extra $U(1)_{FN}$ component.

The lepton mixing data have been undergone a surprising improvement in the last years. From the phenomenological point of view, suitable flavour models should be subject of precise tests. It would be desirable to have some criteria to discard existing models by experiments. For this reason, another goal of the present work is to construct a constrained A_4 model for TB mixing leading to testable sum rules. Many flavour models depend on the smallness of θ_{13} . However, nowadays a sensitivity of a λ_c^2 level is reached only for θ_{12} . Constraining the angles θ_{13} and θ_{23} by a similar sensitivity will require some more years of work. There are two different classes of neutrino flavour models depending on if θ_{13} is not so small, say of $O(\lambda_c)$, or θ_{13} is very small, say of $O(\lambda_c^2)$. Our model belongs to the second class of models. Furthermore, the predicted value of θ_{13} is subjected to the TB

sum rule: $\sin^2\theta_{12} = 1/3 + 2\sqrt{2} (\cos\delta \sin\theta_{13})/3$. We find that this is a general feature of our model: possible deviations from TB mixing are highly constrained giving rise to very stringent next-to-leading predictions.

In the end, we are left with the question whether our model could provide a satisfactory description of the quark sector as well. The most naive way to include quarks is to adopt for them the same classification scheme under $A_4 \times G$ that we have used for leptons. Proceeding in this way, we would obtain an approximately diagonal V_{CKM} . This is a good first order result for quark sector. However, since the mass matrices of up- and down-type quarks belong to the same assignment, one should wonder if they were diagonalized by the same unitary matrix leading to a diagonal V_{CKM} , even including subleading corrections. This is the case, indeed, in a large class of A_4 models [15] and the common opinion is to introduce additional A_4 breaking effects. On the other hand, undesirable sources of A_4 breaking would be dangerous since they could destabilize the desired vacuum alignment. Differently, in our construction, additional sources of A_4 breaking can be easily kept under control and they could play a role in generating a non-trivial V_{CKM} . Nonetheless there is another lack if we were adapt the the same field assignment under $A_4 \times G$ that we have used for leptons also for quarks. The top mass should be generated just at a non-renormalizabel level and this is not a good feature. A detailed analysis of these issues is beyond the scope of this paper and we will leave it for a future work.

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Appendix A

In this appendix we will study the effect of $\chi \sim (1, 1, \omega^2)$ under $A_4 \times Z_3 \times Z'_3$ to the vacuum alignment. All the leading order operators displayed in w_d , see Eq. (15-16), are of dimension three. With the introduction of χ , the driving superpotential should be modified into

$$w_d \rightarrow w_d + \Delta w_d .$$

Given the charge assignment of χ , Δw_d contains only one additional operator up to dimension four:

$$t_1(\varphi_0^T \varphi_T) \chi^2 . \quad (33)$$

After inclusion of this operator, the minimum for φ_T , ξ' , φ_S , ξ , $\tilde{\xi}$ should be shifted. One can look for the new minimum by a perturbation expansion in the power of $1/\Lambda$ around the leading order VEVs given in Eq. (20).

We find that, at the first order in $1/\Lambda$, all the leading order VEVs remain unchanged except for φ_T :

$$\begin{aligned} \langle \varphi_T \rangle &= (\delta v_{T1}, v_T, 0) \quad , \quad \langle \varphi_S \rangle = (v_S, v_S, v_S) \\ \langle \xi \rangle &= u \quad , \quad \langle \tilde{\xi} \rangle = 0 \quad , \quad \langle \xi' \rangle = u' \quad , \quad \langle \chi \rangle = v \quad , \end{aligned} \quad (34)$$

where $\delta v_{T1} = x v^2 / \Lambda$ with $x = t_1 / 4 h_2$ and u, u', v are undetermined. v can slide to a large scale if we introduce a soft mass term with $m_\chi^2 < 0$. However, in order to make sense the previous perturbative analysis, we have to require $v/\Lambda \lesssim \lambda^2$. As expected, a sizable shift $\delta \varphi_{T1}$ can be generated by a relatively large VEV of χ . We will require $v/\Lambda \approx \lambda^2$ and this fixes the order of magnitude of the subleading corrections:

$$\frac{\delta v_{T1}}{\Lambda} \approx \lambda^4 . \quad (35)$$

From this analysis we learn that the order of suppression of δ VEV depends crucially on the abelian factor G and its expected values should be (35) but do not necessarily lie in this range. Different components could in principle receive corrections with different suppressions, even though all leading flavon VEVs are assumed to be of the same order. A more detailed perturbation analysis, in fact, shows that $\delta v_{T3} \sim \lambda^2 \delta v_{T1}$.

To summarize, the introduction of extra singlets perturbs the leading order alignment of φ_T . Due to the presence of the second Z'_3 factor, an internal hierarchy appears in the VEV structure of the triplet

$$\langle \varphi_T \rangle = v_T (\lambda^2, 1, \lambda^4).$$

This is another nice feature of our constrained A_4 model. The previous perturbation analysis shows a simple example in which a non-abelian symmetry can be fully compatible with FN mechanism with abelian charges.

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